

理論的樹高曲線 I

誘導と性質

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論 文

A Theoretical Height-diameter Curve (I)
Derivation and Characteristics

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SWEDA, Tatsuo & UMEMURA, Takeo: A theoretical height-diameter curve (I) Derivation and characteristics J. Jap. For. Soc. 62: 459~464, 1980 Two sets of mathematical formulas for expressing height-diameter relationships of forest stands were derived theoretically. Based on the assumption that both height and diameter growth of individual trees, with respect to time, follow MITSCHERLICH's equation, the first set gives a height-diameter relationship for all-aged stands. The second set is based on the similar but different assumption that individual trees' response to their own respective environmental conditions follows the law of diminishing return, and gives height-diameter relationship for even-aged stands. Both sets of equations are identical in appearance but different in what the coefficients of the equations mean. The significance of the coefficients as well as the applicability and characteristics of the proposed height-diameter curves are discussed.

末田達彦・梅村武夫：理論的樹高曲線（I）誘導と性質 日林誌 62: 459~464, 1980 2種類の理論的樹高曲線を誘導した。第1の樹高曲線は林木の樹高生長および直径生長がともに時間に関するミッチャーリッヒ式に従うと仮定したときに得られるもので、異齢林の樹高曲線を与える。第2の樹高曲線は同齢林のそれを表わすもので、単純同齢林を構成する個々の林木の樹高および直径の生長が、各林木の生育環境条件に対し、収量逓減の法則に従い反応するという仮説より導かれるものである。これら2種類の樹高曲線の数学的表現形式はまったく同じであるが、それぞれの式の意味、したがって係数の生物学的な意味は異なる。

I. Introduction

The relationship between tree height and diameter has been one of the important topics of mensuration largely because of its practical usefulness. Once this relationship is established for a forest stand the time-consuming and still inaccurate height measurements in the field can be replaced by easy estimations from diameters which are relatively easy and fast to determine.

Somewhat subjective but the simplest and most commonly applied method of estimating tree height from diameter, is the free-hand fitting of a height-diameter curve to a set of observations. A more objective method is the least-squares fitting of mathematical equations which relate stem diameter to height in some way or another, and for this purpose numerous mathematical expressions have been presented to date. Most of them give height as one of the following functions of diameter, i. e., either parabolic or logarithmic or exponential. For example:

$$H = a + bD + cD^2 \quad (\text{TROREY, 1932})$$

$$H = a + b(1 - e^{-cD}) \quad (\text{MEYER, 1940})$$

$$H = a + b \log D \quad (\text{MYERS, 1966})$$

where

 H : height D : diameter a, b, c : constants

In addition to these there are literally countless modifications or applications (NISHIZAWA, 1972) so that now it seems almost impossible to decide which one to choose for a specific mensurational purpose. As a matter of fact, all these height-diameter curves are convex upward and show a reasonable degree of fit to observations. It is no wonder, since the goodness of fit has long been the only criterion for adopting new mathematical expressions that they spring up everywhere.

Now it seems to be time to emphasize another important but often unduly ignored criterion, i. e., theoretical reasoning or logical derivation which leads to certain mathematical expressions. As a matter of fact all the above-mentioned height-

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diameter curves are simply empirical or experimental equations with no theoretical reasoning behind them. The only height-diameter curves that carry any theoretical reasoning may be the one advanced by OGAWA *et al.* (1965) and its sophistication by OGINO *et al.* (1967). Based on the assumption that height and diameter satisfy the following modified allometric relationship:

$$\frac{1}{H} \frac{dH}{dt} = h \frac{1}{D} \frac{dD}{dt} \left(\frac{H_{\max} - H}{H_{\max}} \right)$$

where

H : height

D : diameter

$H_{\max}$: maximum height corresponding to

$D = \infty$

t : time

h : allometric coefficient

OGAWA *et al.* derived a height-diameter curve of the form

$$\frac{1}{H} = \frac{1}{AD^h} + \frac{1}{H_{\max}}$$

where

$$\frac{1}{A} = \frac{(H_{\max} - H_0)D_0^h}{H_{\max}H_0}$$

H_0 : minimum height corresponding to the minimum diameter D_0

OGINO's modification consists of incorporating another asymptotic factor.

$$\frac{D_{\max} - D}{D_{\max}}$$

where

$D_{\max}$: maximum diameter

in OGAWA's differential equation given above. This results in a differential equation of the form

$$\frac{1}{H} \frac{dH}{dt} = h \frac{1}{D} \frac{dD}{dt} \left(\frac{H_{\max} - H}{H_{\max}} \right) \left(\frac{D_{\max} - D}{D_{\max}} \right)$$

which upon integration produces a curve of the form

$$\frac{1}{H} = \frac{1}{A'} \left(\frac{D_{\max} - D}{D_{\max}} \right)^h + \frac{1}{H_{\max}}$$

where

$$\frac{1}{A'} = \frac{H_{\max} - H_0}{H_{\max}H_0} \left(\frac{D_0}{D_{\max} - D_0} \right)^h$$

As is often the case with any theoretical work, entirely different lines of reasoning may well be possible. The height-diameter relationship and its theoretical reasoning given in the following sections is one of them. Besides the underlying assumption and mathematical derivation, the applicability of the proposed equations as well as the mensurational

significances of the coefficients appearing in the equations are discussed.

II. Height-diameter Relationship for All-aged Stands

It is acknowledged widely that the growth of individual trees, both in height and diameter, follows MITSCHERLICH's equation (MEYER, 1940; SUZUKI, 1961; NAGUMO and SATO, 1965) of the form

$$\text{height } y = H(1 - e^{-kt}) \quad (1)$$

$$\text{diameter } x = D(1 - e^{-ht}) \quad (2)$$

where

t : time

y : height at time t

x : diameter at time t

H : upper asymptote for height

D : upper asymptote for diameter

k : intrinsic rate of height growth

h : intrinsic rate of diameter growth

e : base of natural logarithm

This fact means that both the height and diameter growths are governed by the following differential equations:

$$\text{height } \frac{dy}{dt} = k(H - y) \quad (3)$$

$$\text{diameter } \frac{dx}{dt} = h(D - x) \quad (4)$$

In other words, Eqs. (1) and (2) are the solutions of Eqs. (3) and (4) respectively.

It is now possible to derive a height-diameter relationship from these equations. Dividing equation (3) by (4) to eliminate the time parameter, we obtain

$$\frac{dy}{dx} = \frac{k(H - y)}{h(D - x)} \quad (5)$$

For the boundary condition we assume that

$$y = y_0 \text{ at } x = x_0 \quad (6)$$

which in terms of tree growth means that the initial height and diameter are equal to y_0 and x_0 respectively at the very beginning of individual tree growth. Separation of variables and subsequent integration with Eq. (5) result in

$$\frac{1}{k} \int_{y_0}^y \frac{1}{H - y} dy = \frac{1}{h} \int_{x_0}^x \frac{1}{D - x} dx$$

the solution of which is given by

$$y = H \left[1 - \frac{H - y_0}{H} \left(\frac{D - x}{D - x_0} \right)^{k/h} \right] \quad (7)$$

This is the general solution for Eq. (5); generally, since no specific mention has been made as to how or at which portion of the stem the height and diameter are to be measured.

If it is the total height and the stem diameter at ground level that is under consideration, then

$$y_0=0 \text{ and } x_0=0 \quad (8)$$

in solution (7). Thus

$$y=H\left[1-\left(\frac{D-x}{D}\right)^{k/h}\right] \quad (9)$$

where

y : total height

x : diameter at ground

results. This is the relationship between the total height and diameter at ground level.

In the ordinary practice of forestry, however, it is the relationship between total height and diameter at breast height (dbh) that is most commonly employed and therefore sought after. To obtain this relationship we put

$$y_0=H_b \text{ and } x_0=0 \quad (10)$$

where

H_b : breast height

i. e., the growth in dbh is initiated just when the tree reaches breast height. Substituting (10) in (7), we get

$$y=H\left[1-L\left(\frac{D-x}{D}\right)^{k/h}\right] \quad (11)$$

where

$$L=\frac{H-H_b}{H}$$

y : total height

x : dbh

It is worth mentioning that the same result can be obtained directly from Eqs. (1) and (2) through arithmetical manipulations. Or more precisely, by solving (2) for time parameter t and substituting in (1), the height-diameter curves (7), (9), or (11) result with an appropriate choice of boundary conditions.

According to the reasoning given so far, Eqs. (7), (9), and (11) represent the relationship between the height and diameter of an individual tree. However, the following assumption or approximation makes these equations also applicable to all-aged stand for their height-diameter curves. Assume an all-aged stand in a steady state where trees of every developmental stage or generation exist, and every generation is in the process of being replaced by the directly succeeding generation, i. e., schematically

seed supply
and
germination

death

...→seedling→young→mature→senescent...→

For this kind of stand, we can assume a mean

height growth curve of form (3) and a mean diameter growth curve of form (4) that represent the growth of trees in every generation on an average basis. It is the process already experienced by the dying trees as well as the course yet to be followed by the seedlings coexisting in the stand. Then as a logical consequence, Eqs. (7), (9), and (11) represent the height-diameter relationship of all-aged stands.

III. Height-diameter Relationship for Even-aged Stands

Tree growth as a function of time is represented by MITSCHERLICH's Eqs. (1) and (2). However, MITSCHERLICH's equation originally was proposed to express plants' response to fertilization and was usually referred to as the law of diminishing return (MITSCHERLICH, 1919). An application of this original reasoning for MITSCHERLICH's equation leads us to individual trees' response in height and diameter growths to their environmental conditions.

MITSCHERLICH's equation originally proposed was of the form

$$y=A(1-e^{-c_1x_1})(1-e^{-c_2x_2})(1-e^{-c_3x_3})\dots$$

where

y : yield

$x_1, x_2, \dots$: amount of factors affecting plant growth

$c_1, c_2, \dots$: intrinsic response coefficients for individual growth factors

A : maximum yield attainable when every growth factor is available with a good surplus

e : base of natural logarithm

In this equation the effect of each growth factor is considered separately and then multiplicatively. However, for the height and diameter growths we consider a single site factor which represents the combined effect of all the conceivable growth factors such as nutrients, moisture, sunlight, etc. Then we get another set of MITSCHERLICH's equations which are exactly the same in appearance as Eqs. (1) and (2) but are different in meaning, i. e.

$$\text{height} \quad y=H'(1-e^{-h'f}) \quad (12)$$

$$\text{diameter} \quad x=D'(1-e^{-h'f}) \quad (13)$$

where

y : height

x : diameter

H', D' : maximum height and diameter attained during a given time interval by an individual tree when it is grown under the most favourable conditions

k', h' : intrinsic response coefficients for height and diameter

f : site index, i.e., a combined effect of numerous growth factors, accumulated for a fixed time interval

It should be noted that here the growth is considered in the alimantal domain, while it was in the time domain in Eqs. (1) and (2). PARKS (1973) argued that growth of animals must be considered in the food-consumption domain rather than in the time domain since the former is more closely related to the growth than the latter. The same argument may well apply to plant growth. It sounds reasonable that plant growth corresponds more closely with the amount of nutrients assimilated and the amount of material photosynthesized during a given time period than with the length of the time period itself. It suffices to mention an often-quoted observation that spruce seedlings suppressed by an overstory for decades show a rapid and vigorous growth once they are exposed to full sunlight when the overstory is removed.

For the derivation of the height-diameter relationship, the same logic as in the preceding case applies. Thus rewriting Eqs. (12) and (13) in differential form as

$$\frac{dy}{df} = k'(H' - y) \quad (14)$$

$$\frac{dx}{df} = h'(D' - x) \quad (15)$$

and dividing (14) by (15), we obtain

$$\frac{dy}{dx} = \frac{k'(H' - y)}{h'(D' - x)} \quad (16)$$

the general solution of which is given by

$$y = H' \left[1 - \frac{H' - y_0}{H'} \left(\frac{D' - x}{D' - x_0} \right)^{k'/h'} \right] \quad (17)$$

where y_0 is the initial height corresponding to the initial diameter x_0 . From this equation, the relationship between the total height and the diameter at ground level is given by

$$y = H' \left[1 - \left(\frac{D' - x}{D'} \right)^{k'/h'} \right] \quad (18)$$

where

y : total height

x : diameter at ground level

The relationship between the total height and dbh is given by

$$y = H' \left[1 - L' \left(\frac{D' - x}{D'} \right)^{k'/h'} \right] \quad (19)$$

where

$$L' = \frac{H' - H_b}{H'}$$

H_b : breast height

y : total height

x : dbh

Obviously, Eqs. (17), (18), and (19) represent the height-diameter relationship of an individual tree. However, similar logic as that in the preceding section makes these equations applicable as height-diameter curves for even-aged stands as follows: Assume the growth up to a certain definite age of an even-aged stand in which the growth of every constituent tree is governed by Eq. (12) in height and by Eq. (13) in diameter. Here, apart from the original meaning, these equations signify the mean growth responses in height and diameter for the stand in question. It is unlikely that all of the trees are governed by exactly the same equations, but this assumption may hold nearly true on an average basis.

Although all of the trees grow under nearly similar conditions, some enjoy more favourable conditions than others depending on the differences in individual site factors and the competition with the surrounding trees. These differences result in the differences in the f values of Eqs. (12) and (13) received by individual trees. For a given f value, there exists a definite and unique height determined by Eq. (12) and a definite and unique diameter determined by Eq. (13) which are interrelated with each other by Eqs. (17), (18), or (19). This relationship holds true for any tree in the stand regardless of the growing conditions it has been subjected to and thus regardless of the f values it has experienced. Thus Eqs. (17), (18), and (19) represent the height-diameter relationship for even-aged stands.

In terms of stand growth, the parameters H' and D' represent the maximum height and diameter to be attained in a specific time period by a few dominant individuals which have been exposed to the most favourable conditions. The other individuals suffering under less favourable conditions for the same period of time take on values smaller than those of the dominants, the degree depending on the severity of individual conditions.

IV. Discussion

As has been noticed already, the proposed height-diameter curves (7), (9), and (11) for all-aged stands and (17), (18), and (19) for even-aged stands do not differ in appearance at all, but they do differ in what they mean. In the former set of equations, the variations in height and diameter are supposed to be attributable to variation in age among the individual trees which constitute an all-aged stand. Thus the asymptotic height H and

diameter D are supposed to be reached by the oldest individuals in the stand, while the younger individuals are of a height somewhere between zero and the asymptote H with a diameter between zero and the asymptote D , depending upon their respective ages. Although in reality, difference both in height and diameter may well exist even among individuals of the same age class, it is assumed to be negligible when compared with the differences among age classes.

In the latter set of equations, however, the variations in height and diameter are supposed to be attributable to the differences in productivity of site on which each tree grows. Since the asymptotic height H' and diameter D' are the maxima attained by the best growing individuals of an even-aged stand by the time the stand reaches a certain age, they increase with the stand age. In other words they are functions of time yet unknown. On the contrary, the asymptotes H and D for all-aged stands are independent of time.

The above argument concerning the applicability of the proposed equations and the corresponding differences in the significance of the coefficients holds both for OGAWA's and OGINO's height-diameter curves. Since both of the equations were derived by eliminating the time parameter, they are good only for all-aged stands. Accordingly, OGAWA (1965) applied his equation to several types of forests, all of which were at their climax stages of succession, and thus were presumably all-aged.

Although OGAWA's and OGINO's original equations are thus limited to all-aged stands they can be modified easily so that they would also apply to even-aged stands as in the present work, i.e. just rewriting in the present work, i.e. just rewriting the original equations in terms of the site factor f instead of time t . It should be noted then that the meanings of the coefficients undergo respective changes.

A graphical representation of total-height *vs.* diameter-at-ground level (18) for even-aged stands is shown in Fig.1 for hypothetical case of $H'=20$ m, $D'=40$ cm, and $k'/h'=1.2, 2.0, 3.0$. Also an example of total height *vs.* dbh (19) for even-aged stands is shown in Fig.2 for another hypothetical case of $H'=20$ m, $H_b'=1.2$ m, $D'=40$ cm, and $k'/h'=1.2, 2.0, 3.0$. It will be readily seen that the proposed height-diameter curves are convex upward in agreement with general observations as well as with most of the empirical equations. It can be noticed also that the convexity increases as the k'/h' ratio increase.

In forestry management and planning, tree height

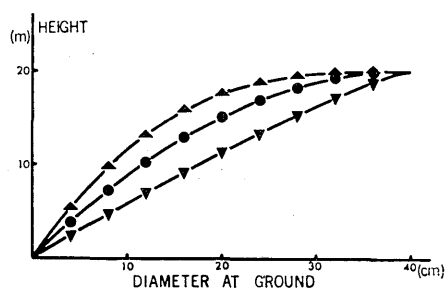


Fig. 1. Total height against diameter at ground level as expressed by the proposed Eq. (18)

$H'=20$ m, $D'=40$ cm, $k'/h'=1.2$ (∇), 2.0 ($\bullet$), 3.0 ($\blacktriangle$)

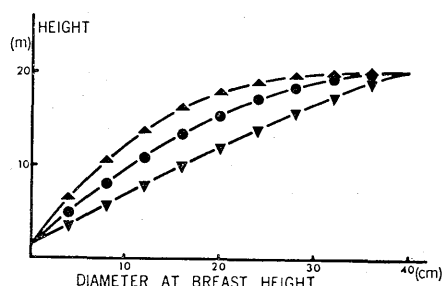


Fig. 2. Total height against diameter at breast height as expressed by the proposed Eq. (19)

$H'=20$ m, $H_b'=1.2$ m, $D'=40$ cm, $k'/h'=1.2$ (∇), 2.0 ($\bullet$), 3.0 ($\blacktriangle$)

at a certain age, for example 50 years, is often used as an index of the site's productivity, i.e. site index. This common practice is based on a silvicultural rule of thumb that the tree height responds more quickly and sensitively to the site's productivity than does the diameter. This in terms of MITSCHERLICH's Eqs. (12) and (13) means that intrinsic response coefficient k' for height growth is greater than coefficient h' for diameter growth. Thus the ratio k'/h' in Eqs. (18) and (19) is usually greater than unity. This results in the upward convexity of the proposed height-diameter curves. The proposed equations produce curves convex downward if the k'/h' ratio is smaller than unity. However, this in reality is not likely.

The same argument on the shape of the curve applies to the height-diameter relationship (7), (9), and (11) for all-aged stands. The quicker response of the height growth to the site's productivity than that of diameter is just another manifestation of the fact that the height growth has a greater intrinsic rate than the diameter growth, i.e.

$$k > h$$

Thus the ratio k/h is greater than unity, and this eventually ends up with upward convexity of the height-diameter curves (7), (9), and (11). It is interesting to note that the k/h ratio is nearly equal to unity for open-growing individual trees (KOBAYASHI, 1978). Although KOBAYASHI's sample was small, being of size three, this fact suggests a close relationship between a stands' stem density and the k/h ratio.

V. Conclusion

In this work, the emphasis was placed on the derivation of a set of height-diameter curves as well as on the theoretical reasoning underlying the derivation. Also the discussion of the applicability of the resultant equations was made from a theoretical point of view rather than from practical one. For the proposed equations to be functional in practice, further research is needed on their practical characteristics and feasibility. This should include: i) an investigation into the goodness of fit to actual data, especially in comparison with the other existing height-diameter curves, either empirical or theoretical, and ii) the determination of the numerical ranges of parameters, particularly the k/h ratio, and their relationship with different types of forest stands.

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